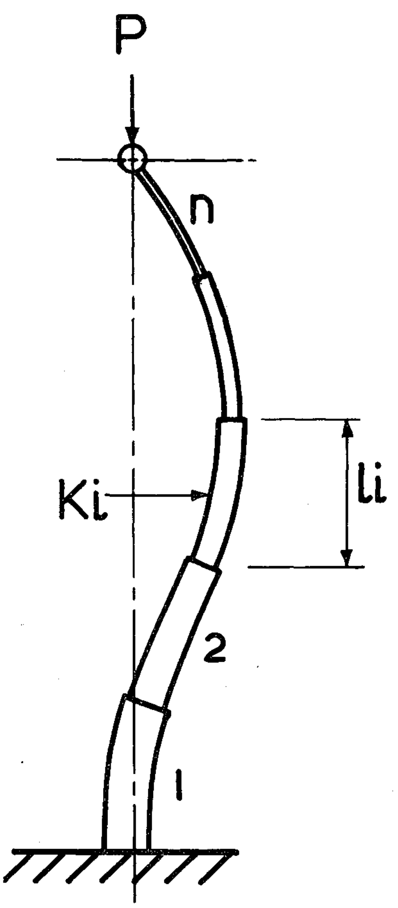

FIG. 2.

REFERENCE

[1] Timoshenko, S., *Theory of Elastic Stability*, p. 88, McGraw-Hill Book Co., New York, 1936.

A Formula for Certain Types of Stiffness Matrices of Structural Elements

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September 24, 1962

THE LINEAR THEORY is assumed throughout. The procedure is outlined for the three-dimensional case. Let

$$\left.\begin{aligned} k &= \mathrm{col}\,(k_1, k_2, \ldots, k_m) \\ \sigma &= \mathrm{col}\,(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) \\ \epsilon &= \mathrm{col}\,(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}) \end{aligned}\right\} \qquad (1)$$

where k is a vector of constants, σ the stress vector and ϵ the strain vector at some point P in the element. The stress and strain vectors are related by an equation of the form

$$\epsilon = N\sigma \qquad (2)$$

where N is a nonsingular symmetric matrixi nvolving Poisson's ratio and Young's modulus. We let

$$\sigma = Uk \qquad (3)$$

where U is a matrix whose elements are of the form $x^{\nu_1}y^{\nu_2}z^{\nu_3}$ with the ν_i integers or zero, so that each stress is assumed approximated by a polynomial whose coefficients are components of the vector k. The polynomials are chosen to satisfy the differential equations of equilibrium and compatibility.

Following somewhat the general plan of Refs. 3 and 4 the stresses are integrated over the surface of the element giving a relation between the applied loads p, the resultants of these stresses, and the vector k.

$$p = Vk \qquad (4)$$

Since the stresses satisfy the conditions of equilibrium the element is also in equilibrium under the loads p. The case discussed here is special in that the applied loads are assumed to determine the vector k and hence the stresses and conversely. Hence rank $(V) = m$.

For convenience of notation the first m rows of V are assumed linearly independent, hence a partitioning of Eq. (4) gives

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{V_1}{V_2}\right) k \qquad (5)$$

where V_1 is nonsingular. From this one obtains

$$p_1 = V_1 k \qquad p_2 = V_2 k \qquad (6)$$

giving

$$k = V_1^{-1} p_1 \qquad (7)$$

Hence p_1 alone determines k and the components of p_1 may be taken as independent variables. Fixing the displacements corresponding to p_2 the element becomes supported.

Now consider the flexibility matrix of the supported element. In Ref. 1 the following formula is given:

$$f_{i,j} = \int_V \epsilon^{iT} \sigma_j dV \qquad (8)$$

where ϵ^i = the column strain vector at point P in the element of volume V due to a unit load at coordinate i. ϵ^{iT} = the transpose of ϵ^i, σ_j = the column stress vector at P due to a unit load at j. Combining Eq. (3) with (7) and letting e_j = a column vector with 1.0 in the jth place and zeros elsewhere gives

$$\sigma^j = UV_1^{-1}e_j \qquad (9)$$

from which by use of Eq. 2 one obtains

$$\epsilon^{iT} = (N\sigma^i)^T = \sigma^{iT}N = e_i^T V_1^{-1T} U^T N \qquad (10)$$

Inserting in Eq. (8) and noting that only the components of the U matrix are affected by the integration gives

$$f_{i,j} = e_i^T V_1^{-1T} \left(\int_V U^T N U dV\right) V_1^{-1} e_j = e_i^T V_1^{-1T} G V_1^{-1} e_j \qquad (11)$$

where

$$G = \int_V U^T N U dV \qquad (12)$$

But (11) amounts to the matrix equation

$$F = (f_{i,j}) = V_1^{-1T} G V_1^{-1} \qquad (13)$$

We now show that the matrix G is nonsingular. The energy stored under any loading is, by Ref. 2,

$$U = {}^1/_2 \int_V \epsilon^T \sigma dV \qquad (14)$$

which by use of Eqs. 2 and 3 becomes

$$U = {}^1/_2 \int_V \sigma^T N \sigma dV = {}^1/_2 \int_V k^T U^T N U k dV = {}^1/_2 k^T G k \qquad (15)$$

Now stored energy can be zero only if the loading is zero, which implies that the vector k is also zero. Otherwise it is positive. This amounts to saying that G is positive-definite and hence nonsingular.

By the inversion of Eq. (13) one obtains the $m \times m$ stiffness matrix of the supported element

$$\bar{S} = V_1 G^{-1} V_1^T \qquad (16)$$

To obtain the unsupported $n \times n$ stiffness matrix S we define the load transformation matrix H to satisfy

$$\left(\frac{p_1}{p_2}\right) = H p_1 \qquad (17)$$

so that

$$S = H\bar{S}H^T \tag{18}$$

From Eqs. (6) and (7) we see that

$$H = (V_2{}^I V_1{}^{-1}) \tag{19}$$

where I is an identity matrix of mth order. Inserting this in Eq. (18) and recalling Eq. (4) gives, finally, the simple formula

$$S = VG^{-1}V^T \tag{20}$$

for the unsupported stiffness matrix of the structural element.

The matrix G can be conveniently evaluated by numerical integration using a formula of the type

$$G = \sum_i \gamma_i U_i^T N U_i \tag{21}$$

where the γ_i are constants and U_i = the value of U at some point i.

References

[1] Argyris, J. H., *Energy Theorems and Structural Analysis, Part 1, General Theory*, Aircraft Engineering, Eq. (84), p. 43, Feb. 1955.

[2] Timoshenko, S., and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill Book Co., Inc., 2nd Ed., p. 147, Eq. (c), 1951.

[3] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., *Stiffness and Deflection Analysis of Complex Structures*, Journal of the Aeronautical Sciences, Vol. 23, No. 9, Sept., 1956.

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On Minor-Circle Turns

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With the advent of the space age, many people have become concerned with the maneuvering characteristics of space vehicles. A great deal of literature is available on the various aspects of orbital transfers and on the various methods of providing improved performance. In a recent paper W. H. T. Loh[1] introduced a definition of a minor-circle turn. We would like to suggest at this time an alternative definition which appears useful in that it removes some of the restrictions found in the work by Loh.

Loh's definition of a minor-circle turn requires the vehicle to fly in a plane which is oriented normal to the axis of the earth and which is elevated above the equator at that distance where the plane intersects the surface of the earth at the desired latitude. While this is satisfactory for near-earth trajectories and for gentle minor-circle turns, it does not appear to be capable of handling the cases where the altitudes are large and where the latitudes of the turn approach ninety degrees. To surmount this difficulty, let us define a minor-circle trajectory as that path where the vector pointing toward the center of the earth from the vehicle always cuts the surface of the earth at a constant latitude. This is equivalent to requiring the path of the vehicle to lie on the surface of a cone whose solid angle is the supplement of twice the latitude (see Fig. 1). With this definition, we are no longer restricted to near-earth trajectories or to gentle turns but may include sharp turns with steep re-entry angles without violating our assumptions. The equations of motion may be written by inspection if we choose our three directions as normal to the surface of the cone, tangent to the velocity vector lying in the cone, and perpendicular to these two directions. Defining γ as our ascent angle, α as our bank angle measured out of the cone, and λ as our latitude, we obtain

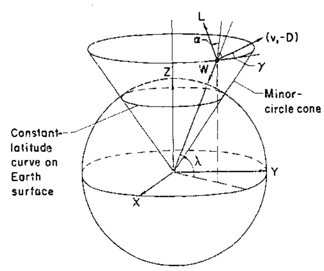

Fig. 1. Minor-circle cone.

$$(1/2)m(dv^2/ds) = m(dv/dt) = -D - mg \sin \gamma$$

$$mv^2(d\gamma/ds) = L \cos \alpha - mg \cos \gamma + (mv^2/r) \cos \gamma$$

$$(mv^2/r) \cos^2 \gamma \tan \lambda = L \sin \alpha$$

where s is the distance along the path.

These three equations may be compared to those of Loh and found to be similar but not identical. For near-earth orbits of slowly varying altitudes, the equivalence is clear as is expected from the geometrical considerations of the two definitions. It is also clear that this new definition adds no new restrictions and thus the results obtained by Loh in his paper may be reproduced if desired. To demonstrate this, his Eq. (2) becomes, with our definition and notation,

$$\frac{dv^2}{ds} + \frac{2g}{(L/D)} \sqrt{\cos^2 \gamma \left\{ \left[\frac{v^2}{gr} \cos \gamma \tan \lambda \right]^2 + \left[1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma} \left(\frac{d\gamma}{ds} \right) \right]^2 \right\}} = -2g \sin \gamma$$

and the aerodynamic control required at any moment along the minor circle becomes

$$\sin \alpha = \frac{(v^2/gr) \cos \gamma \tan \lambda}{\sqrt{\left[\frac{v^2}{gr} \cos \gamma \tan \lambda \right]^2 + \left[1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma} \left(\frac{d\gamma}{ds} \right) \right]^2}}$$

which is similar to Eq. (3) in Loh's paper.

Reference

[1] Loh, W. H. T., *Dynamics and Thermodynamics of Re-Entry*, Journal of Aerospace Sciences, Vol. 27, No. 10, Oct. 1960.

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Design of Tufts for Flow Visualization

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Tufts are frequently used for visualizing gas or liquid motions near solid surfaces. They are especially useful in develop-